B.Sc. Part–II (Semester–IV) Examination

MATHEMATICS

MODERN ALGEBRA GROUPS AND RINGS

Time : 3 Hours] [Maximum Marks : 60

Note :– (1) Question No. **1** is compulsory and attempt it once only.

(2) Solve **one** question from each unit.

1. Choose the correct alternative (1 mark each) :

(i) Let G be a group and let $a \in G$ if $O(a) = 3$ then $O(a^{-1})$ is equal to :

(a) 0 (b) 1

(c) 2 (d) 3

(ii) Every subgroup of a cyclic group is :

(a) Non-abelian (b) Cyclic

(c) Cyclic but not abelian (d) Abelian but not Cyclic

(iii) A group having only improper normal subgroup is called :

(a) A finite group (b) A permutation group

(c) A simple group (d) None of these

(iv) The identity element of a quotient group G/H is :

(a) G (b) H

(c) H/G (d) G/H

(v) if ϕ is homomorphism of a group G onto G' with kernel K then G' is :

(a) Isomorphic to G/K (b) Isomorphic to K/G

(c) Isomorphic to G (d) One-one homomorphism

(vi) A homomorphism of a group into itself is :

(a) A homomorphism (b) An isomorphism

(c) An endomorphism (d) None of these

(vii) The characteristic of an integral domain is :

(a) Even number (b) Odd number

(c) Prime number (d) None of these

(viii) An integral domain is :

(a) Always a field (b) Never a field

(c) A field when it is finite (d) A field when it is infinite

(ix) A field which contains no proper subfield is called :

(a) Prime field (b) Subfield

(c) Integral domain (d) Division ring

(x) A ring which has only trivial ideal is called :

(a) Prime Ring (b) Commutative Ring

(c) Division Ring (d) Simple Ring 10×1=10

UNIT—I

2\. (a) Show that if G is an abelian group then $(ab)^n = a^n b^n$ $\forall a, b, \in G$ & $\forall$ integer n. 5

(b) Define even permutation and for $S = \{1, 2, 3, ..., 9\}$ and $a, b, \in A(s)$, compute $a^{-1}ba$ where $a = (5, 7, 9)$ and $b = (1, 2, 3)$. 1+4

3\. (p) Prove that union of two subgroups is subgroup if one is contained in the other. 4

(q) Show that cube root of unity from an abelian group with respect to the usual multiplication of numbers. 3

(r) Show that if every element of the group g is its own inverse then G is abelian. 3

UNIT–II

4\. (a) Prove that the subgroup N of G is a normal subgroup of G if and only if each left coset of N in G is a right coset of N in G. 4

(b) Let $G = \{1, -1, i, -i\}$ and $N = \{1, -1\}$ Show that N is normal subgroup of the multiplicative group G. Find the quotient group G/N and find its identity. 4

(c) Show that if G is an abelian then the quotient group is also abelian. 2

5\. (p) Suppose that N and M are two normal subgroups of G and that $N \cap M = \{e\}$. Show that for any $n \in N, m \in M, nm = mn$. 4

(q) Let H be a subgroup of G. If $N(H) = \{g \in G \mid gHg^{-1} = H\}$ then show that N(H) is subgroup of G. 4

(r) Show that the intersection of two normal subgroups of G is a normal subgroup of G. 2

UNIT–III

6\. (a) Define homomorphism. If $\phi : G \to G'$ is a homomorphism then show that :

(i) $\phi(e) = e'$

(ii) $\phi(x^{-1}) = [\phi(x)]^{-1}$ $\forall x \in G$

where e and e' be the identity in G and G' resp. 1+4

(b) If M and N are normal subgroups of G then prove that $\frac{NM}{M} \cong \frac{N}{N \cap M}$. 5

7\. (p) Show that if $\phi : G \to G'$ is homomorphism with kernel K, then K is normal subgroup of G. 5

(q) Let G be any group. g is a fixed element in G. If $\phi : G \to G'$ defined by $\phi(x) = g x g^{-1}$ then prove that ϕ is an isomorphism of G onto G. 5

UNIT–IV

8. (a) Define :

    (i) Integral Domain

    (ii) Field

    Prove that a field is an integral domain but converse is not true. 5

   (b) Prove that the characteristic of an integral domain is either zero or prime number. 5

9. (p) Define :

    (i) Ring with unity

    (ii) Without zero divisor

    (iii) Prime field. 3

   (q) Prove that intersection of two subrings is a subring. 2

   (r) Prove that a non-empty subset K of a field F, is a subfield of F if and only if $x - y, xy^{-1} \in K$, $y \neq 0 \; \forall \; x, y \in K$. 5

UNIT–V

10. (a) If R is commutative ring with unit element whose only ideal {0} & R then prove that R is a field. 5

    (b) If U is an ideal of a ring R then prove that R/U is a ring. 5

11. (p) If U and V are ideals of a ring R then prove that

    (i) $U \cap V$ is an ideal of R

    (ii) $U \cap V$ is the largest ideal that is contained in both U and V. 5

    (q) Define :

    (i) Maximal ideal

    (ii) Principal ideal

    (iii) Prime Ideal. 3

    (r) If U is an ideal of a ring R with unity 1 and $1 \in U$ prove that U = R. 2